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A new subclass of *H*-matrices

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ABSTRACT

Class of *H*-matrices plays an important role in various scientific disciplines, in economics, for example. However, this class could be used in order to get various benefits in other linear algebra fields, like determinant estimation, Perron root estimation, eigenvalue localization, improvement of convergence area of relaxation methods, etc. For that reason, it seems important to find a subclass of *H*-matrices, as wide as possible, and expressed by explicit conditions, involving matrix elements only. One step forward in this direction, starting from Gudkov matrices, from one side, and *S*-SDD matrices, from the other side, will be presented in this paper.

1. Introduction

We know the relationships between the following classes of matrices: strictly diagonally dominant matrices (SDD), Ostrowski matrices, *S*-strictly diagonally dominant matrices (*S*-SDD) and *H*-matrices. Namely, each of mentioned classes is a proper subset of the next one. We also know that the same relations hold for SDD matrices, Ostrowski matrices, Gudkov matrices [3], and *H*-matrices. But, the class of Gudkov matrices stands in general position with the class of *S*-SDD matrices, as we will show in this paper, using simple examples. The goal of this paper is to generalize both classes – Gudkov matrices, as well as *S*-SDD matrices.

Throughout the paper we will use the following notations:

$N := \{1, 2, \ldots, n\}$ for the set of indices

S for any nonempty subset of N

$\overline{S} := N \setminus S$ for the complement of S

$r_i(A) := \sum_{k\in N, k\neq i} |a_{ik}|$ for ith row sum and $r_i^S(A) := \sum_{k\in S, k\neq i} |a_{ik}|$ for part of ith row sum,

which corresponds to the subset S.

We define $h_i(A)$ recursively:

$$h_1(A) := \sum_{j\neq 1} |a_{1j}|,$$

$$h_i(A) := \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=i+1}^{n} |a_{ij}|,$$

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and $h_i^S(A)$:

$$h_1^S(A) := r_1^S(A),$$

$$h_i^S(A) := \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j^S(A)}{|a_{jj}|} + \sum_{j=i+1, j\in S}^{n} |a_{ij}|.$$

Obviously, for arbitrary subset S and each index $i \in N$,

$$r_i(A) = r_i^S(A) + r_i^{\bar{S}}(A) \quad \text{and} \quad h_i(A) = h_i^S(A) + h_i^{\bar{S}}(A).$$

The following definition and theorem are well-known.

Definition 1. A matrix $A = [a_{ij}] \in \mathbf{C}^{n,n}$ is called an H-matrix if its comparison matrix $\langle A \rangle = [m_{ij}]$ defined by

$$m_{ii} = |a_{ii}|, \quad m_{ij} = -|a_{ij}|, \quad i,j = 1, 2, \ldots, n, \; i \neq j$$

is an M-matrix, i.e., $\langle A \rangle^{-1} \geqslant 0$.

Theorem 1. *A matrix $A = [a_{ij}] \in \mathbf{C}^{n,n}$ is an H-matrix if and only if there exists a diagonal nonsingular matrix W such that AW is an SDD matrix. Moreover, we can always assume that W has only positive diagonal entries.*

2. Known subclasses of *H*-matrices

Definition 2. A matrix $A = [a_{ij}] \in \mathbf{C}^{n,n}$ is called SDD matrix if, for each $i \in N$, it holds that

$$|a_{ii}| > r_i(A).$$

Definition 3 (see [5]). A matrix $A = [a_{i,j}] \in \mathbf{C}^{n,n}, n \geqslant 2$, is called Ostrowski matrix if

$$|a_{ii}| \cdot |a_{jj}| > r_i(A) \cdot r_j(A), \quad \text{for all } i \neq j, \; i,j \in N.$$

Definition 4 (see [1]). Given any matrix $A = [a_{ij}] \in \mathbf{C}^{n,n}, n \geqslant 2$, and given any nonempty proper subset S of N, then A is an S-strictly diagonally dominant (S-SDD) matrix if

$$|a_{ii}| > r_i^S(A) \quad \text{for all } i \in S \quad \text{and}$$

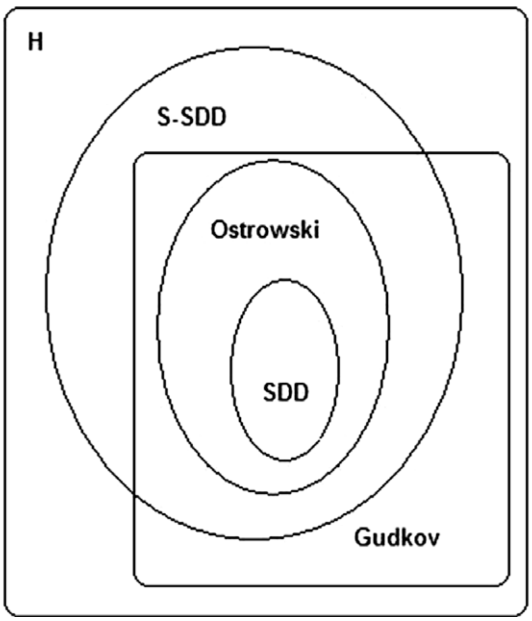
$$(|a_{ii}| - r_i^S(A))(|a_{jj}| - r_j^{\bar{S}}(A)) > r_i^{\bar{S}}(A) r_j^S(A) \quad \text{for all } i \in S, \; j \in \bar{S}.$$

Definition 5. A matrix $A = [a_{ij}] \in \mathbf{C}^{n,n}, n \geqslant 2$ is called Nekrasov matrix if, for each $i \in N$, it holds that

$$|a_{ii}| > h_i(A).$$

If there exists a permutation matrix P, such that PAP^T is a Nekrasov matrix, then we will say that A is a Gudkov matrix.

It is known, see, for example, [2,4], that above subclasses of H-matrices stand in the following position:



3. A new subclass of H-matrices

Definition 6. Given any matrix $A = [a_{i,j}] \in \mathbf{C}^{n,n}, n \geqslant 2$, and given any nonempty proper subset S of N, then A is an S-Nekrasov matrix if

$$|a_{ii}| > h_i^S(A) \quad \text{for all } i \in S,$$
$$|a_{jj}| > h_j^{\overline{S}}(A) \quad \text{for all } j \in \overline{S} \quad \text{and}$$
$$(|a_{ii}| - h_i^S(A))(|a_{jj}| - h_j^{\overline{S}}(A)) > h_i^{\overline{S}}(A)h_j^S(A) \quad \text{for all } i \in S,\ j \in \overline{S}.$$

Definition 7. Given any matrix $A = [a_{i,j}] \in \mathbf{C}^{n,n}, n \geqslant 2$, and given any nonempty proper subset S of N, then A is an S-Gudkov matrix if there exists a permutation matrix P, such that PAP^T is an S-Nekrasov matrix.

Theorem 2. *If a matrix $A \in \mathbf{C}^{n,n}$ is an S-Gudkov matrix, then A is nonsingular, moreover it is an H-matrix.*

Proof. It is sufficient to prove that each S-Nekrasov matrix is an H-matrix. In order to do that, for arbitrary nonempty proper set of indices S, let us define the interval $J_A(S)$ as

$$J_A(S) := (\mu_1^S(A), \mu_2^S(A)),$$
$$\mu_1(A) := \max_{i \in S} \frac{h_i^{\overline{S}}(A)}{|a_{ii}| - h_i^S(A)} \quad \text{and} \quad \mu_2(A) := \min_{j \in \overline{S}, h_j^S(A) \neq 0} \frac{|a_{jj}| - h_j^{\overline{S}}(A)}{h_j^S(A)}.$$

Since matrix A is S-Nekrasov matrix, we know that:

$$|a_{ii}| > h_i^S(A) \quad \text{for all } i \in S, \tag{1}$$
$$|a_{jj}| > h_j^{\overline{S}}(A) \quad \text{for all } j \in \overline{S} \quad \text{and} \tag{2}$$
$$(|a_{ii}| - h_i^S(A))(|a_{jj}| - h_j^{\overline{S}}(A)) > h_i^{\overline{S}}(A)h_j^S(A) \quad \text{for all } i \in S,\ j \in \overline{S}, \tag{3}$$

which implies:

$$\frac{h_i^{\overline{S}}(A)}{|a_{ii}| - h_i^S(A)} < \frac{|a_{jj}| - h_j^{\overline{S}}(A)}{h_j^S(A)} \quad \text{for all } i \in S,\ j \in \overline{S} \quad \text{and} \quad h_j^S(A) \neq 0.$$

Then, obviously, the interval $J_A(S)$ is nonempty, so we can choose $\gamma \in J_A(S)$, and define the diagonal matrix W in the following way:

$$W = \operatorname{diag}(w_1, w_2, \ldots, w_n),$$

where

$$w_i = \begin{cases} \gamma > 0, & i \in S, \\ 1, & i \in \overline{S}. \end{cases}$$

We will show that matrix AW is a Nekrasov matrix, and, hence, nonsingular. In other words, we will prove that matrix AW satisfies the following inequality:

$$|(AW)_{ii}| > h_i(AW) \quad \text{for all } i \in N.$$

First, let us prove by induction that

$$h_i(AW) = \gamma h_i^S(A) + h_i^{\overline{S}} \quad \text{for all } i \in N. \tag{4}$$

For $i = 1$ we have

$$h_1(AW) = \sum_{j \neq 1} |(AW)_{1j}| = \sum_{j \neq 1} |a_{1j} w_j| = \gamma \sum_{j \in S, j \neq 1} |a_{1j}| + \sum_{j \in \overline{S}, j \neq 1} |a_{1j}| = \gamma r_1^S(A) + r_1^{\overline{S}}(A) = \gamma h_1^S(A) + h_1^{\overline{S}}(A).$$

Assume that equality (4) is satisfied for all $i < k$, and prove that it is satisfied for $i = k$:

$$\begin{aligned} h_k(AW) &= \sum_{j=1}^{k-1} |(AW)_{kj}| \frac{h_j(AW)}{|(AW)_{jj}|} + \sum_{j=k+1}^{n} |(AW)_{kj}| = \sum_{j=1}^{k-1} |a_{kj}||w_j| \frac{h_j(AW)}{|a_{jj}||w_j|} + \sum_{j=k+1}^{n} |a_{kj}||w_j| \\ &= \sum_{j=1}^{k-1} |a_{kj}| \frac{\gamma h_j^S(A) + h_j^{\overline{S}}}{|a_{jj}|} + \sum_{j=k+1, j \in S}^{n} \gamma |a_{kj}| + \sum_{j=k+1, j \in \overline{S}}^{n} |a_{kj}| \\ &= \gamma \sum_{j=1}^{k-1} |a_{kj}| \frac{h_j^S(A)}{|a_{jj}|} + \gamma \sum_{j=k+1, j \in S}^{n} |a_{kj}| + \sum_{j=1}^{k-1} |a_{kj}| \frac{h_j^{\overline{S}}(A)}{|a_{jj}|} + \sum_{j=k+1, j \in \overline{S}}^{n} |a_{kj}| = \gamma h_k^S + h_k^{\overline{S}}. \end{aligned}$$

Because $\gamma \in J_A(S)$, we know

$$\frac{h_i^{\overline{S}}(A)}{|a_{ii}| - h_i^S(A)} < \gamma < \frac{|a_{jj}| - h_j^{\overline{S}}(A)}{h_j^S(A)} \quad \text{for all } i \in S,\ j \in \overline{S} \quad \text{and} \quad h_j^S(A) \neq 0,$$

which implies that

$$\gamma|a_{ii}| > \gamma h_i^S + h_i^{\overline{S}} \quad \text{for all } i \in S \quad \text{and}$$

$$|a_{jj}| > \gamma h_j^S + h_j^{\overline{S}} \quad \text{for all } j \in \overline{S}.$$

Then we have

$$|(AW)_{ii}| = \gamma|a_{ii}| > \gamma h_i^S + h_i^{\overline{S}} = h_i(AW) \quad \text{for all } i \in S \quad \text{and}$$

$$|(AW)_{jj}| = |a_{jj}| > \gamma h_j^S + h_j^{\overline{S}} = h_j(AW) \quad \text{for all } j \in \overline{S},$$

which means

$$|(AW)_{ii}| > h_i(AW) \quad \text{for all } i \in N.$$

Hence, matrix AW is a Nekrasov matrix, and it is nonsingular, thus matrix A is also nonsingular.

To prove that A is an H-matrix, it is sufficient to recall that Nekrasov matrices are H-matrices, therefore they can be diagonally scaled to SDD form. More precisely, since AW is a Nekrasov matrix, there exists a diagonal nonsingular matrix X, such that AWX is an SDD matrix. Then, according to Theorem 1, A is an H-matrix. □

4. Relationships between subclasses of *H*-matrices

First, we are going to show that neither of the classes, Gudkov and S-SDD matrices, is subset of the other one. In order to do that, we will use two simple examples.

Example 1. Matrix

$$A = \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0.5 \\ 1 & 2 & 0.33 & 1 \end{bmatrix},$$

is S-SDD matrix for $S = \{1, 2\}$, and it is not a Gudkov matrix.

Example 2. Matrix

$$B = \begin{bmatrix} 1 & 0.5 & 0 & 0.4 \\ 0.5 & 1 & 0.5 & 0 \\ 0 & 0.5 & 1 & 0.5 \\ 0.5 & 0 & 0.5 & 1 \end{bmatrix},$$

is a Gudkov matrix, and it is not an S-SDD matrix for neither one subset S.

Finally, we will prove that the class of S-Gudkov matrices contains both subclasses: S-SDD matrices and Gudkov matrices.

It is easy to see that each Gudkov matrix is an S-Gudkov matrix, for arbitrary subset S of indices.

To verify that the class of S-SDD matrices is a subset of the class of S-Gudkov matrices, it is sufficient to show that each S-SDD matrix is an S-Nekrasov matrix, too. Indeed, if A is an S-SDD matrix, then there exists a diagonal nonsingular matrix

$$W = \mathrm{diag}(w_1, w_2, \ldots, w_n),$$

where

$$w_i = \begin{cases} \gamma > 0, & i \in S, \\ 1, & i \in \overline{S}, \end{cases}$$

such that AW is an SDD matrix. Since each SDD matrix is Nekrasov one, it follows that AW is a Nekrasov matrix, i.e.

$$|(AW)_{ii}| > h_i(AW) \quad \text{for all } i \in N.$$

As we have already seen, it means that

$$\gamma|a_{ii}| > \gamma h_i^S + h_i^{\overline{S}} \quad \text{for all } i \in S \quad \text{and}$$

$$|a_{jj}| > \gamma h_j^S + h_j^{\overline{S}} \quad \text{for all } j \in \overline{S},$$

or, equivalently,

$$\frac{h_i^{\overline{S}}(A)}{|a_{ii}| - h_i^S(A)} < \gamma < \frac{|a_{jj}| - h_j^{\overline{S}}(A)}{h_j^S(A)} \quad \text{for all } i \in S,\ j \in \overline{S} \quad \text{and} \quad h_j^S(A) \neq 0.$$

Then, obviously, all three conditions

$$|a_{ii}| > h_i^S(A) \quad \text{for all } i \in S,$$
$$|a_{jj}| > h_j^{\overline{S}}(A) \quad \text{for all } j \in \overline{S} \quad \text{and}$$
$$(|a_{ii}| - h_i^S(A))(|a_{jj}| - h_j^{\overline{S}}(A)) > h_i^{\overline{S}}(A) h_j^S(A) \quad \text{for all } i \in S,\ j \in \overline{S},$$

from the definition of S-Nekrasov matrices, are satisfied.

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