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Geršgorin-type localizations of generalized eigenvalues

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SUMMARY

We introduce several localization techniques for the generalized eigenvalues of a matrix pair, obtained via the famous Geršgorin theorem and its generalizations. Specifically, we address the techniques of computing and graphing of the obtained localization sets of a matrix pair. The work that follows involves much about nonnegative matrices, strictly diagonally dominant (SDD) matrices, H- and M-matrices. We show the utility of our results theoretically, as well as with numerical examples and graphs.

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1. INTRODUCTION

We start by introducing the basic concepts of this paper. Given matrices $A, B \in \mathbb{C}^{n,n}$, with $n \geqslant 1$, the family of matrices $A - zB$, parameterized by the complex number z, is called a *matrix pencil*. Then, whenever we refer to a matrix pencil $A - zB$ or to a matrix pair (A, B), we are speaking about the same object.

A matrix pair (A, B) is called *regular* if $\det(A - zB)$ is not identically zero, and otherwise it is called *singular*. If the matrix pair (A, B) is regular, then

$$\det(A - zB) =: p(z) \tag{1}$$

where $p(z)$ is a polynomial in z, which is of degree at most n. If $\lambda \in \mathbb{C}$ is such that $p(\lambda) = 0$, i.e. $\det(A - \lambda B) = 0$, then λ is called a *generalized eigenvalue* of the matrix pair (A, B), and there

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exists a nonzero vector $\mathbf{v}\in\mathbb{C}^n$ such that $A\mathbf{v}=\lambda B\mathbf{v}$, where $\mathbf{v}$ is called a *generalized eigenvector* of the matrix pair (A,B). The collection of all eigenvalues of a matrix pair (A,B) (also called the *spectrum* of the matrix pencil) is denoted by

$$\sigma(A,B):=\{z\in\mathbb{C}:\det(A-zB)=0\} \tag{2}$$

Clearly, if $B=I_n\in\mathbb{C}^{n,n}$, then the spectrum of the matrix pair (A,B) reduces to the standard spectrum of A, i.e. $\sigma(A,I_n)=\sigma(A)$. Also, if B is nonsingular, it is easy to see that $0=\det(A-zB)=\det(B^{-1}A-zI_n)$, so that in this case, the spectrum of the matrix pair (A,B) reduces to the spectrum of $B^{-1}A$.

It is also known that the degree of the polynomial $p(z)$, in (1), is n if and only if B is nonsingular. This implies that if B is singular, then $p(z)$ is of degree r with $r<n$, so the number of the generalized eigenvalues of the matrix pair (A,B) is r, and, by convention, the remaining $n-r$ eigenvalues are set equal to ∞.

Here, our object is to *estimate* the spectra of regular matrix pencils, much as the union of the Geršgorin disks of a given matrix $A\in\mathbb{C}^{n,n}$ *estimates* the eigenvalues of A.

We begin with the well-known results of Lévy–Desplanques and Geršgorin (cf. [1], Theorems 1.1 and 1.3).

Theorem 1 (*Lévy–Desplanques*)
Let $A=[a_{ij}]\in\mathbb{C}^{n,n}$, with $n\geqslant 2$, be such that

$$|a_{ii}|>r_i(A) \quad \text{for each } i\in N:=\{1,2,\ldots,n\} \tag{3}$$

where $r_i(A)$ is defined to be the ith deleted absolute row sum of A, i.e.

$$r_i(A):=\sum_{j\in N\backslash\{i\}}|a_{ij}| \quad (\text{any } i\in N) \tag{4}$$

Then, A is a nonsingular matrix.

The matrices that fulfill condition (3) are known in the literature as *strictly diagonally dominant matrices* (further denoted as SDD matrices).

Theorem 2 (*Geršgorin*)
Let $A=[a_{ij}]\in\mathbb{C}^{n,n}$, with $n\geqslant 2$, and let λ be an eigenvalue of A. Then, there exists an index $i\in N$ such that $|\lambda-a_{ii}|\leqslant r_i(A)$. Thus, with $\sigma(A)$ denoting the spectrum of the matrix A, we have that

$$\sigma(A)\subset\Gamma(A):=\bigcup_{i\in N}\Gamma_i(A) \tag{5}$$

where

$$\Gamma_i(A):=\{z\in\mathbb{C}:|z-a_{ii}|\leqslant r_i(A)\} \quad (\text{any } i\in N) \tag{6}$$

The set $\Gamma(A)$ from (5) is called the *Geršgorin set* for the matrix A, while $\Gamma_i(A)$ of (6) is called the ith *Geršgorn disk* for A.

What was emphasized in [1] was that the nonsingularity result of Theorem 1 is exactly *equivalent* to the eigenvalue inclusion result of Theorem 2.

To carry this a step further, it is also known from [1] that the next nonsingularity result of Olga Taussky in Theorem 3 is *equivalent* to her result of Theorem 4.

Theorem 3 (*Taussky* [2])
Let $A=[a_{ij}]\in\mathbb{C}^{n,n}$ be an irreducibly diagonally dominant matrix, i.e. A is irreducible (cf. [1], p.11) and

$$\begin{aligned}&|a_{ii}|\geqslant r_i(A) \quad \text{for each } i\in N\\ &\text{with strict inequality holding for at least one} \quad i\in N\end{aligned} \tag{7}$$

Then, A is nonsingular.

Theorem 4 (*Taussky* [3])
Let $A=[a_{ij}]\in\mathbb{C}^{n,n}$ be irreducible. If $\lambda\in\sigma(A)$ is such that $\lambda\notin\operatorname{int}\Gamma_i(A)$ for any $i\in N$, i.e. $|\lambda-a_{ii}|\geqslant r_i(A)$ for each $i\in N$, then

$$|\lambda-a_{ii}|=r_i(A) \quad \text{for each } i\in N \tag{8}$$

i.e. all the Geršgorin circles $\{z\in\mathbb{C}:|z-a_{ii}|=r_i(A)\}$ pass through λ.

As before, the equivalence of Theorems 3 and 4 can also be readily verified.

There is a relatively new *third* equivalence, due to Cvetković [4], which plays a vital role in our study here of the generalized eigenvalue problem. We begin with the following definitions of M-matrices and H-matrices, due to Ostrowski in 1937. These are stated in [1, 5], Appendix C.

There are many equivalent definitions of an M-matrix, given in Berman and Plemmons [5], but the following one is best suited for our needs here. Given a real $n\times n$ matrix A having the form $A=sI-B$, where B, in $\mathbb{R}^{n,n}$, has all nonnegative entries, let $\rho(B):=\max\{|\lambda|:\lambda\in\sigma(B)\}$, where $\rho(B)$ denotes the *spectral radius* of B. Then, the matrix A is called an *M-matrix* if $\rho(B)\leqslant s$, and A is called a *nonsingular M-matrix* if $\rho(B)<s$.

Next, given a complex matrix $A=[a_{ij}]\in\mathbb{C}^{n,n}$, then the matrix $\langle A\rangle:=[m_{ij}]\in\mathbb{R}^{n,n}$, called the *comparison matrix* for A, is defined, for all $1\leqslant i,j\leqslant n$, by

$$m_{ij}:=\begin{cases}|a_{ii}|, & i=j\\ -|a_{ij}|, & i\neq j\end{cases} \tag{9}$$

With this definition of a comparison matrix, then $A=[a_{ij}]\in\mathbb{C}^{n,n}$ is called an *H-matrix* if its comparison matrix $\langle A\rangle$ is an M-matrix, and similarly, A is called a *nonsingular H-matrix*, if $\langle A\rangle$ is a nonsingular M-matrix.

In particular, we next utilize the following known connection (see [5], Theorem 2.3) between nonsingular H-matrices and SDD matrices.

Theorem 5
A matrix $A=[a_{ij}]\in\mathbb{C}^{n,n}$ is a nonsingular H-matrix if and only if there exists a nonsingular diagonal matrix $X=\operatorname{diag}[x_1,x_2,\ldots,x_n]$ such that AX is a SDD matrix, i.e. an SDD matrix.

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Now, let the vector $\mathbf{x}=[x_1,x_2,\ldots,x_n]^{\mathrm{T}}\in\mathbb{R}^n$ has all entries positive, and set $X:=\mathrm{diag}[x_1, x_2,\ldots,x_n]$. Then,

$$r_i^{\mathbf{x}}(A):=x_i^{-1}\sum_{j\in N\backslash\{i\}}|a_{ij}|x_j \quad (\text{any } i\in N) \tag{10}$$

denotes the *ith weighted deleted absolute row sum of* A, and we define the corresponding disks and localization sets as

$$\Gamma_i^{r^{\mathbf{x}}}(A):=\{z\in\mathbb{C}:|z-a_{ii}|\leqslant r_i^{\mathbf{x}}(A)\} \quad (\text{any } i\in N) \tag{11}$$

and

$$\Gamma^{r^{\mathbf{x}}}(A):=\bigcup_{i\in N}\Gamma_i^{r^{\mathbf{x}}}(A) \tag{12}$$

By intersecting over all positive vectors $\mathbf{x}\in\mathbb{R}^n$, we obtain the localization set

$$\Gamma^{\mathfrak{R}}(A):=\bigcap_{\mathbf{x}>\mathbf{0}}\Gamma^{r^{\mathbf{x}}}(A) \tag{13}$$

called the *minimal Geršgorin set*. This was first introduced in [6], and it was extensively investigated in [1, 7]. Then, as in the pattern above, the nonsingularity result of Theorem 5 (more precisely, its 'if' part) turns out to be *equivalent* to the following eigenvalue inclusion result, which directly follows from the definition of the minimal Geršgorin set.

Theorem 6
Let $A=[a_{ij}]\in\mathbb{C}^{n,n}$, and let $\Gamma^{\mathfrak{R}}(A)$ of (13) be the minimal Geršgorin set for A. If $\lambda\in\sigma(A)$, then $\lambda\in\Gamma^{\mathfrak{R}}(A)$; thus, $\sigma(A)\subseteq\Gamma^{\mathfrak{R}}(A)$.

This equivalence was pointed out in [4], and it gives us a starting point in obtaining localization results for generalized eigenvalues. In particular, we remark that the points of the spectrum of a given matrix A can be considered as complex points z for which the matrix pencil $A-zI$ *loses* nonsingularity. Now, as we have seen from the above introduction, points of the Geršgorin set of a matrix A can be characterized as the points where the matrix pencil $A-zI$ loses its SDD property. This brings us to the construction of a generalized Geršgorin set in Section 2. Construction of the generalized minimal Geršgorin set is done similarly in Section 3. In each section, in addition, we present useful properties, analyze computations, and give examples for the constructed localization areas.

2. THE GENERALIZED GERŠGORIN SET

In the following sections, if not stated otherwise, we suppose that $A,B\in\mathbb{C}^{n,n}$, with $n\geqslant 2$, and that the matrix pair (A,B) is *regular*.

Definition 1
We define the set $\Gamma(A,B)$ as:

$$\Gamma(A,B):=\{z\in\mathbb{C}:A-zB \text{ is not an SDD matrix}\} \tag{14}$$

and this set is called the (*generalized*) Geršgorin set of a matrix pair (A,B).

On replacing the phrase 'not an SDD' in (14) with the term 'singular', the generalized Geršgorin set of a matrix pair (A, B) becomes the (generalized) spectrum of a matrix pair (A, B), given in (2). From this observation, the proof of the next theorem is clear.

Theorem 7
Given matrices $A, B \in \mathbb{C}^{n,n}$, with $n \geqslant 2$, the generalized spectrum of the matrix pair (A, B) belongs to the Geršgorin set of the matrix pair (A, B), i.e. the following inclusion holds:

$$\sigma(A, B) \subset \Gamma(A, B) \tag{15}$$

It is easy to see from (14) that $\Gamma(A, B) = \bigcup_{i \in N} \Gamma_i(A, B)$, where

$$\Gamma_i(A, B) := \left\{ z \in \mathbb{C} : |b_{ii}z - a_{ii}| \leqslant \sum_{j \in N \setminus \{i\}} |b_{i,j}z - a_{ij}| \right\} \quad (\text{all } i \in N) \tag{16}$$

Moreover, if we denote, by $r_i^S(A) := \sum_{j \in S \setminus \{i\}} |a_{ij}|$, that part of a row sum that corresponds to the columns given by the set of indices $S \subseteq N$, and if we denote the particular sets of indices $\beta(i) := \{j \in N : b_{ij} \neq 0\}$ and $\overline{\beta}(i) := \{j \in N : b_{ij} = 0\}$, for all $i \in N$, then we can write

$$\Gamma_i(A, B) = \left\{ z \in \mathbb{C} : \left| z - \frac{a_{ii}}{b_{ii}} \right| |b_{ii}| - \sum_{j \in \beta(i) \setminus \{i\}} \left| z - \frac{a_{ij}}{b_{ij}} \right| |b_{ij}| \leqslant r_i^{\overline{\beta}(i)}(A) \right\} \tag{17}$$

whenever $i \in \beta(i)$, and otherwise, we can write

$$\Gamma_i(A, B) = \left\{ z \in \mathbb{C} : |a_{ii}| - r_i^{\overline{\beta}(i)}(A) \leqslant \sum_{j \in \beta(i)} \left| z - \frac{a_{ij}}{b_{ij}} \right| |b_{ij}| \right\} \tag{18}$$

We remark that a set $\Gamma_i(A, B)$, as defined in (16), can be either the *empty set* or the entire complex plane $\mathbb{C}$, which can occur when $\beta_i(A) = \emptyset$, i.e. when all entries of the ith row of the matrix B are zero. Then, the ith generalized Geršgorin set has the following form:

$$\Gamma_i(A, B) = \{z \in \mathbb{C} : |a_{ii}| \leqslant r_i^{\overline{\beta}(i)}(A) = r_i(A)\} \tag{19}$$

Thus,

$$\Gamma_i(A, B) = \begin{cases} \emptyset & \text{if } |a_{ii}| > r_i(A) \\ \mathbb{C} & \text{if } |a_{ii}| \leqslant r_i(A) \end{cases} \tag{20}$$

Of course, when the second case of (20) occurs, the matrix B is singular, and $p(z) = \det(A - zB)$ has degree less then n. As we are considering regular matrix pencils, the degree of the polynomial $p(z)$ has to be at least one; thus, at least one of the sets $\Gamma_i(A, B)$ has to be nonempty, implying that the generalized Geršgorin set of a regular matrix pencil is always nonempty.

On inspecting the form of the generalized Geršgorin 'disks' of (17) and (18), when this second case of (20) occurs, we can establish the following properties.

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Theorem 8
Let $A, B \in \mathbb{C}^{n,n}$, with $n \geqslant 2$. Then, the following statements hold:

1. Let $i \in N$ be such that for at least one $j \in N$, $b_{i,j} \neq 0$. Then, the ith generalized Geršgorin set, $\Gamma_i(A, B)$, as defined in (17) and (18), is an unbounded set in the complex plane $\mathbb{C}$ if and only if $|b_{ii}| \leqslant r_i(B)$, where $r_i(B)$ is defined from (4).
2. The generalized Geršgorin set $\Gamma(A, B)$ is a compact set in $\mathbb{C}$ if and only if B is an SDD matrix.
3. The ith generalized Geršgorin set $\Gamma_i(A, B)$, given in (16), contains zero if and only if $|a_{ii}| \leqslant r_i(A)$.
4. The generalized Geršgorin set $\Gamma(A, B)$ contains zero if and only if A is not an SDD matrix.
5. If there exists an $i \in N$ such that both $b_{ii} = 0$ and $|a_{ii}| \leqslant r_i^{\overline{\beta}(i)}(A)$, then $\Gamma_i(A, B)$, and consequently $\Gamma(A, B)$, is the entire complex plane.

Proof
First, it is evident that 2. and 4. follow directly from 1. and 3. respectively. Moreover, 3. is easy to obtain, by putting $z = 0$ in the inequalities of (16), and 5. follows directly from (18). Then, it remains to prove 1.

If $i \in N \setminus \beta(i)$, then clearly $|b_{ii}| = 0 \leqslant r_i(B)$ and $\Gamma_i(A, B)$ is unbounded from (16). Thus, let $i \in \beta(i)$. First, let us suppose that $\Gamma_i(A, B)$ is unbounded. Then, there is a sequence $\{z_k\}_{k \in \mathbb{N}}$ of complex numbers such that $|z_k| \to \infty$, as $k \to \infty$, and $z_k \in \Gamma_i(A, B)$. Then, for a sufficiently large $k \in \mathbb{N}$, we have

$$|z_k|(|b_{ii}| - r_i(B)) \leqslant r_i^{\overline{\beta}(i)}(A) \tag{21}$$

Now, if $|b_{ii}| > r_i(B)$, then taking the limit as $k \to \infty$ in (21), we obtain a contradiction. Conversely, let $|b_{ii}| \leqslant r_i(B)$, and let $\{z_k\}_{k \in \mathbb{N}}$ be a sequence of complex numbers such that $|z_k| \to \infty$, when $k \to \infty$. Then, it is easy to see that, for a sufficiently large $k \in \mathbb{N}$,

$$\left| z - \frac{a_{ii}}{b_{ii}} \right| |b_{ii}| - \sum_{j \in \beta(i) \setminus \{i\}} \left| z - \frac{a_{ij}}{b_{ij}} \right| |b_{ij}| \leqslant 0$$

and thus, $z_k \in \Gamma_i(A, B)$ from (18). □

Example 1
Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 0.2 \\ 0 & -1 & 0.4 & 0 \\ 0 & 0 & i & 1 \\ 0.2 & 0 & 0 & -i \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0.5 & 0.1 & 0.1 & 0.1 \\ 0 & -1 & 0.1 & 0.1 \\ 0 & 0 & i & 0.1 \\ 0.1 & 0 & 0 & -0.5i \end{pmatrix}$$

By inspection, B is an SDD matrix and, according to the item 2 of Theorem 8, the set $\Gamma(A, B)$ is compact in the complex plane. This shows Figure 1, where the actual generalized eigenvalues are marked with little circles.

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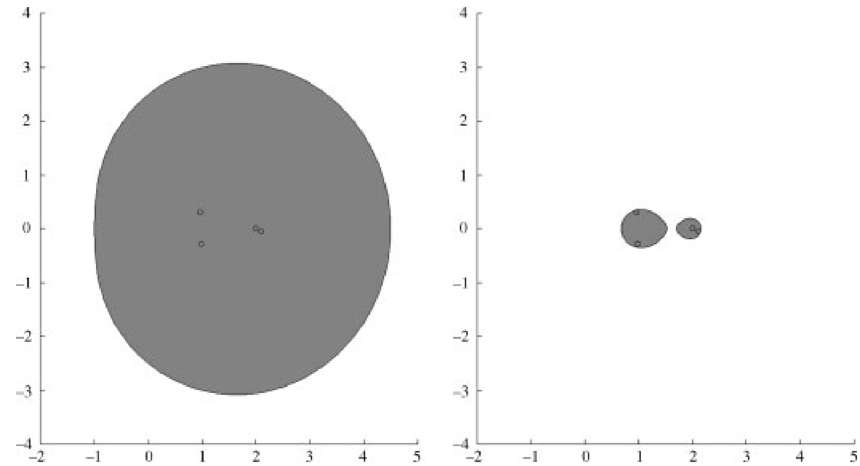

Figure 1. Generalized Geršgorin set of the matrix pair (A, B) of the Example 1 on the left, and the corresponding generalized minimal Geršgorin set (Definition 2) on the right.

3. GENERALIZED MINIMAL GERŠGORIN SET

We begin, as in the previous section, with

Definition 2
The set $\Gamma^{\Re}(A, B)$, defined as

$$\Gamma^{\Re}(A, B) := \{z \in \mathbb{C} : A - zB \text{ is not a nonsingular } H\text{-matrix}\}$$

is called the generalized minimal Geršgorin set of the matrix pair (A, B).

This time we have weakened the singularity property of a matrix pencil, in the point z, to be the property that $A - zB$ is not a nonsingular H-matrix, in order to 'enlarge' spectrum up to the generalized minimal Geršgorin set. Then, from (2) and Definition 2, we have

Theorem 9
Given matrices $A, B \in \mathbb{C}^{n,n}$, with $n \geqslant 2$, the generalized spectrum of the matrix pair (A, B) belongs to the generalized minimal Geršgorin set of the matrix pair (A, B), i.e. the following inclusion holds:

$$\sigma(A, B) \subset \Gamma^{\Re}(A, B) \tag{22}$$

3.1. Basic properties

From Theorem 5, it is evident that, for any nonsingular diagonal matrix X,

$$\Gamma^{\Re}(A, B) := \bigcap_{\mathbf{x} > \mathbf{0}} \Gamma(X^{-1}AX, X^{-1}BX)$$

and, as before, we can obtain the following analog of Theorem 8.

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Theorem 10
Let $A, B \in \mathbb{C}^{n,n}$ with $n \geqslant 2$. Then, the following statements hold:

1. The generalized minimal Geršgorin set $\Gamma^{\mathfrak{R}}(A, B)$ is compact in $\mathbb{C}$ if and only if B is a nonsingular H-matrix.
2. The generalized minimal Geršgorin set $\Gamma^{\mathfrak{R}}(A, B)$ contains zero if and only if A is not a nonsingular H-matrix.
3. If there exists $i \in N$ such that $b_{ii} = 0$, then $a_{ii} = 0$ if and only if $\Gamma^{\mathfrak{R}}(A, B) = \mathbb{C}$.

Let us now address the problem of computing and plotting the generalized minimal Gešgorin set for a given matrix pair. First, we remark that, in the case of the original minimal Geršgorin set, progress has recently been made in computing its tight approximation with an iterative approach (see [7]). Here, we will develop an analogue of this, in the sense of our generalized minimal Geršgorin set. Thus, we will need the necessary tools, derived from the Perron–Frobenius theory of nonnegative matrices.

For a given matrix pencil $A - zB \in \mathbb{C}^{n,n}$ and a given $z \in \mathbb{C}$, we define matrix the $Q_z := -\langle A - zB \rangle$, where the comparison matrix operator $\langle \cdot \rangle$ is defined in (9). Defining $\delta(z) := \max\{|a_{ii} - zb_{ii}| : i \in N\}$, and putting $P_z := Q_z + \delta(z) I$, we obtain the nonnegative matrix P_z which, by the Perron–Frobenius theory of nonnegative matrices [5], possesses a real, nonnegative eigenvalue $\rho(P_z)$, called the *Perron root* of P_z.

Now, by setting $v(z) := \rho(P_z) - \delta(z)$,we have from Theorem C.2 in [1] that

$$v(z) = \inf_{\mathbf{x} > \mathbf{0}} \left\{ \max_{i \in N} [(Q_z \mathbf{x})_i / x_i] \right\} \tag{23}$$

or equivalently,

$$v(z) = \inf_{\mathbf{x} > \mathbf{0}} \left\{ \max_{i \in N} \left[x_i^{-1} \cdot \sum_{j \in N \backslash \{i\}} |b_{ij} z - a_{ij}| x_j - |b_{ii} z - a_{ii}| \right] \right\} \tag{24}$$

Thus, the following characterization of the generalized minimal Geršgorin set holds.

Theorem 11
Given any two matrices $A, B \in \mathbb{C}^{n,n}$, with $n \geqslant 2$, then

$$z \in \Gamma^{\mathfrak{R}}(A, B) \quad \text{if and only if } v(z) \geqslant 0 \tag{25}$$

The proof of this theorem follows in the same way as in the proof of the Proposition 4.3 of [1], which characterizes the minimal Geršgorin set of (13). In addition, following the argument of [7], the real-valued complex function v is continuous, and the generalized minimal Geršgorin set is a closed set in the extended complex plain $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$, and we similarly obtain that

$$z \in \partial \Gamma^{\mathfrak{R}}(A, B) \quad \text{if and only if} \begin{cases} \text{(i)} & v(z) = 0 \quad \text{and} \\ \text{(ii)} & \text{there exists a sequence of complex} \\ & \text{numbers } \{z_j\}_{j=1}^{\infty} \text{ such that } \lim\limits_{j \to \infty} z_j = z \\ & \text{and } v(z_j) < 0 \text{ for all } j \geqslant 1 \end{cases} \tag{26}$$

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This brings us to notion of a *star-shaped* set, needed in the next result. A set S in $\mathbb{C}$ is said to be a star-shaped with respect to a given point z_0, if for every z in S, the entire line segment $\{\alpha z_0+(1-\alpha)z : 0\leqslant\alpha\leqslant 1\}$ lies in S.

Theorem 12
For any two matrices $A, B\in\mathbb{C}^{n,n}$, with $n\geqslant 2$, such that B is a nonsingular H-matrix, then $v(a_{kk}/b_{kk})\geqslant 0$ for each $k\in N$. Moreover, for each $k\in N$ and for each θ with $0\leqslant\theta\leqslant 2\pi$, there exists an $\hat{\varrho}_k(\theta)\geqslant 0$ such that the entire complex interval $[(a_{kk}/b_{kk})+t\mathrm{e}^{\mathrm{i}\theta}]_{t=0}^{\hat{\varrho}_k(\theta)}$ is contained in $\Gamma^{\mathfrak{R}}(A,B)$, and, consequently, the set

$$\bigcup_{\theta=0}^{2\pi}\left[\frac{a_{kk}}{b_{kk}}+t\mathrm{e}^{\mathrm{i}\theta}\right]_{t=0}^{\hat{\varrho}_k(\theta)} \tag{27}$$

is a subset of $\Gamma^{\mathfrak{R}}(A,B)$, which we call the *star-shaped* subset of $\Gamma^{\mathfrak{R}}(A,B)$, with respect to the point a_{kk}/b_{kk}.

Proof
As B is a nonsingular H-matrix, then, for every $k\in N$, $b_{kk}\neq 0$, and, by putting $z=a_{kk}/b_{kk}$ in (24), we obtain

$$v\left(\frac{a_{kk}}{b_{kk}}\right)\geqslant\inf_{\mathbf{x}>\mathbf{0}}\left\{x_k^{-1}\cdot\sum_{j\in N\backslash\{k\}}\left|\frac{a_{kk}b_{kj}}{b_{kk}}-a_{kj}\right|x_j\right\}\geqslant 0$$

Thus, a_{kk}/b_{kk} lies in the set $\Gamma^{\mathfrak{R}}(A,B)$. Now, for a fixed θ in $0\leqslant\theta\leqslant 2\pi$, consider the ray $[(a_{kk}/b_{kk})+t\mathrm{e}^{\mathrm{i}\theta}]$, $t\geqslant 0$. Its starting point lies in $\Gamma^{\mathfrak{R}}(A,B)$, which is, according to Theorem 10, a compact set in $\mathbb{C}$. Thus, there exists a point $(a_{kk}/b_{kk})+\hat{\varrho}_k(\theta)\mathrm{e}^{\mathrm{i}\theta}$, which lies on the boundary of the $\Gamma^{\mathfrak{R}}(A,B)$. Taking the smallest $\hat{\varrho}_k(\theta)$ of such points fulfills the conditions of the theorem. □

Now, for a fixed θ with $0\leqslant\theta\leqslant 2\pi$, it is interesting to note that if $v(a_{kk}/b_{kk})=0$ and if $\hat{\varrho}_k(\theta)=0$, then a_{kk}/b_{kk} actually lies on the boundary of $\Gamma^{\mathfrak{R}}(A,B)$. In addition, if $\hat{\varrho}_k(\theta)=0$ for each θ with $0\leqslant\theta\leqslant 2\pi$, then a_{kk}/b_{kk} is a generalized eigenvalue of the pair (A,B). This brings us to

Theorem 13
Given two matrices $A, B\in\mathbb{C}^{n,n}$, with $n\geqslant 2$, for which there exists a $k\in N$ such that $b_{kk}=0$, then for every sequence of complex numbers $\{z_k\}_{k=1}^{\infty}$ such that $|z_k|\to\infty$, as $k\to\infty$, there exists an $\alpha\geqslant 0$ such that $v(z_k)\to\alpha$, and, by our convention, we write $v(\infty)>0$. Moreover, if A is a nonsingular H-matrix, then, for each θ with $0\leqslant\theta\leqslant 2\pi$, there exists a $\hat{\varrho}_k(\theta)>0$ such that the whole complex interval $[\hat{\varrho}_k(\theta)\mathrm{e}^{\mathrm{i}\theta}+t]_{t=0}^{\infty}$ is contained in $\Gamma^{\mathfrak{R}}(A,B)$, and, consequently, the set

$$\bigcup_{\theta=0}^{2\pi}[\hat{\varrho}_k(\theta)\mathrm{e}^{\mathrm{i}\theta}+t]_{t=0}^{\infty} \tag{28}$$

is a subset of $\Gamma^{\mathfrak{R}}(A,B)$, which we will call a *star-shaped* subset of $\Gamma^{\mathfrak{R}}(A,B)$ with respect to ∞.

Proof
The idea of this proof is the following. For any $z\neq 0$ in $\sigma(A,B)$, then from (2) we have that $\det(A-zB)=0$, from which it follows that $\det(B-(1/z)A)=0$. This necessarily implies that

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$1/z\in\sigma(B,A)$, which we can interpret as

$$\sigma(B,A)=\frac{1}{\sigma(A,B)}$$

and similarly

$$\Gamma^{\mathfrak{R}}(B,A)=\frac{1}{\Gamma^{\mathfrak{R}}(A,B)}$$

Here, for an arbitrary set S of complex numbers (including infinity), i.e. $S\subset\mathbb{C}_\infty$, we define the set $1/S$ as

$$\frac{1}{S}=\left\{\frac{1}{z}:z\in S\right\}$$

With this, for a $k\in N$ such that $b_{kk}=0$ and $a_{kk}\neq 0$, we have that the star-shaped set of (27) for the matrix pair (B,A), corresponding to the center $b_{kk}/a_{kk}=0$, transforms to the set (28). □

3.2. Computing the generalized minimal Geršgorin set

Having the properties of the previous subsection, the problem of graphing the generalized minimal Geršgorin set becomes the problem of graphing the subset of $\mathbb{C}_\infty$, for which the function $v(z)$ is nonnegative. In order to resolve this problem, we need to find a way to compute the value of the function $v(z)$, for different values of z. For the case when the concept of *irreducibility* (see [5]) is involved, we have the following result.

Theorem 14
Given the matrices $A,B\in\mathbb{C}^{n,n}$ with $n\geqslant 2$, let the matrix pencil $A-zB$, at the point $z\in\mathbb{C}$, be irreducible. Then, for each $\mathbf{x}>\mathbf{0}$ in $\mathbb{R}^n$, either

$$\min_{i\in N}\{(Q_z\mathbf{x})_i/x_i\}<v(z)<\max_{i\in N}\{(Q_z\mathbf{x})_i/x_i\} \tag{29}$$

or

$$Q_z\mathbf{x}=v(z)\mathbf{x} \tag{30}$$

As (29) and (30) suggest, we can use the power method as a tool to compute the eigenvalue $v(z)$ in the following way. We start with the nonnegative matrix P_z, which we assume to be *irreducible*. Then, either P_z is primitive or it can be shifted to a primitive matrix $P_z+\varepsilon I$, $\varepsilon>0$, (see Section 2.2 of [8]). Thus, either way, we can apply power iterations to compute $\rho(P_z)$.

Starting with an $\mathbf{x}^{(0)}>\mathbf{0}$ in $\mathbb{R}^n$, the power iteration gives convergent upper and lower estimates for $\rho(P_z)$, i.e. if $\mathbf{x}^{(m)}:=P_z^m\mathbf{x}^{(0)}$ for all $m\geqslant 1$, then with $\mathbf{x}^{(m)}:=[x_1^{(m)},x_2^{(m)},\ldots,x_n^{(m)}]^{\mathrm{T}}$, we have that

$$\underline{\lambda_m}:=\min_{i\in N}\left\{\frac{x_i^{(m+1)}}{x_i^{(m)}}\right\}\leqslant\rho(P_z)\leqslant\max_{i\in N}\left\{\frac{x_i^{(m+1)}}{x_i^{(m)}}\right\}=:\overline{\lambda_m} \tag{31}$$

for all $m\geqslant 1$, and

$$\lim_{m\to\infty}\underline{\lambda_m}=\rho(P_z)=\lim_{m\to\infty}\overline{\lambda_m} \tag{32}$$

Thus,

$$\underline{\lambda_m} - \delta(z) \leqslant v(z) \leqslant \overline{\lambda_m} - \delta(z)$$

It is important to say that, from (31), we do not need to find the value of $v(z)$ with great accuracy, and it is sufficient to iterate until either one of the next two conditions is fulfilled:

1. $\underline{\lambda_m} > \delta(z)$, implying that $v(z) > 0$ and, thus, $z \in \Gamma^{\mathfrak{R}}(A, B)$, or
2. $\overline{\lambda_m} < \delta(z)$, implying that $v(z) < 0$ and, thus, $z \in \mathbb{C}_\infty \setminus \Gamma^{\mathfrak{R}}(A, B)$.

If neither one is fulfilled until we achieve a certain accuracy $\varepsilon > 0$, i.e. $\overline{\lambda_m} - \underline{\lambda_m} < \varepsilon$, we conclude that z lies in the ε-neighborhood of a boundary point of $\Gamma^{\mathfrak{R}}(A, B)$.

So, the simplest way to plot an approximation of the generalized minimal Geršgorin set $\Gamma^{\mathfrak{R}}(A, B)$ is to introduce the coarse grid, say $n_x \times n_y$, of the $[-L, L]^2 \subset \mathbb{C}$, for sufficiently large $L > 0$. For this grid, we will have $n_x n_y$ complex nodes and, we determine, for each node, which category each node belongs. Each of them will be either 'colored' to be in the $\Gamma^{\mathfrak{R}}(A, B)$, if either $\underline{\lambda_m} > \delta(z)$ (case 1) is the case or $\overline{\lambda_m} - \underline{\lambda_m} < \varepsilon$, where ε represents the coarseness of the grid. If $\underline{\lambda_m} < \delta(z)$ (case 2) occurs, the point is left 'uncolored', as it is in the exterior of the $\Gamma^{\mathfrak{R}}(A, B)$.

Example 2
Let

$$M_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & i & 1 \\ 0.7 & 0 & 0 & -i \end{pmatrix}, \quad N_1 = \begin{pmatrix} 1 & 0.1 & 0 & 0 \\ 0.1 & 1 & 0.1 & 0 \\ 0 & 0.1 & 1 & 0.1 \\ 0 & 0 & 0.1 & 1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & 1 & 0 & 0.5 \\ 0 & -1 & 0.5 & 0 \\ 0 & 0 & i & 1 \\ 1 & 0 & 0 & -i \end{pmatrix}, \quad N_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0.5 & 0 & 0 & 1 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 0.5 & 0 & 0 & 0.3 \\ 0 & 0.5 & 0.1 & 0 \\ 0 & 0 & 0.7 & 0.1 \\ 1.2 & 0 & 0 & 0.7 \end{pmatrix} \quad \text{and} \quad N_3 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & i & 1 \\ 2 & 0 & 0 & -2i \end{pmatrix}$$

In Figure 2, a few examples of generalized minimal Geršgorin sets are plotted, using this simple approach. From the top left corner to the right bottom corner, the inclusion regions $\Gamma^{\mathfrak{R}}(M_1, N_1)$, $\Gamma^{\mathfrak{R}}(M_2, N_2)$, $\Gamma^{\mathfrak{R}}(M_3, N_3)$ and $\Gamma^{\mathfrak{R}}(N_3, M_1)$ are colored, and the actual generalized eigenvalues are marked by little circles.

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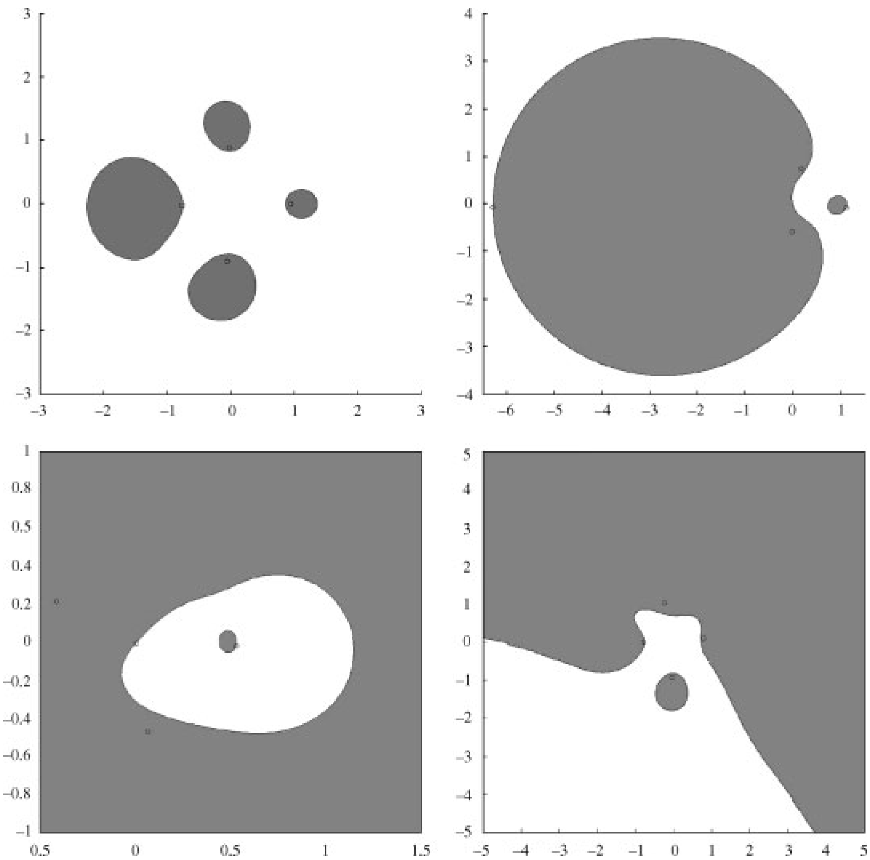

Figure 2. Examples of the generalized minimal Geršgorin sets of 4×4 matrices.

Another way to plot the generalized minimal Geršgorin set is to compute its approximation. A way to do it is a modification of the approach presented in [7]. Here, this approach could be useful when H-matrices are involved, as the result of Theorem 12 gives the main motivation.

Namely, we are interested in determining the star-shaped subsets of the generalized minimal Geršgorin set. Thus, assuming that B is a nonsingular H-matrix and that $A-zB$ is an irreducible matrix for each $z\in\mathbb{C}$, we start by fixing an index $k\in N$, and the corresponding center a_{kk}/b_{kk} of the star-shaped subset of (27). For each $\theta\in[0,2\pi]$, since $v(a_{kk}/b_{kk})>0$, we can, with a few trial steps, find Δ, $\Delta>0$, such that $v((a_{kk}/b_{kk})+\Delta)<0$. Then, we can apply the bisection search to the interval $[a_{kk}/b_{kk},(a_{kk}/b_{kk})+\Delta]$ to determine $\hat{\varrho}_k(\theta)$. As a result, we have approximated the boundary point $(a_{kk}/b_{kk})+\hat{\varrho}_k(\theta)e^{i\theta}$ of the generalized minimal Geršgorin set.

Now, moving the angles $\theta\in[0,2\pi]$, we obtain the approximation of the set of (27).

How this is used in graphing the generalized minimal Geršgorin set for the matrix pair of Example 1, can be seen in Figure 3, where the center $a_{11}/b_{11}=2$ is marked with a small square, the actual generalized eigenvalues are again marked by little circles, and each of the obtained approximations of the boundary points is marked with a dot. In this example, it was sufficient to consider only one star-shaped subset of $\Gamma^{\mathfrak{R}}(A,B)$, but this is not always the case, as Figure 2 implies.

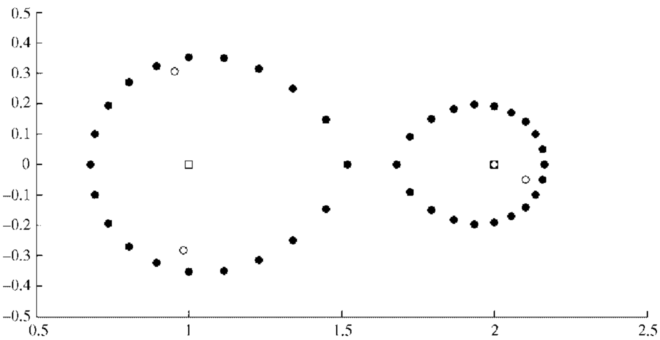

Figure 3. Approximation of the generalized minimal Geršgorin set of Example 1.

3.3. The Generalized minimal Geršgorin set is, in a sense, minimal

As the generalized minimal Geršgorin set is obtained as a intersection of all 'weighted' generalized Geršgorin sets, it is, in a sense, the minimal area that contains the generalized spectrum. On the other hand, we see from (17) and (18) that the generalized Geršgorin set is defined uniquely from the following data: $|a_{ij}|, |b_{ij}|$ and a_{ij}/b_{ij}, when $b_{ij}\neq 0$, where $i, j\in N$. Thus, for every matrix pair that leaves this data set unchanged, its generalized spectrum will be included in the same generalized Geršgorin set, and consequently in the generalized minimal Geršgorin set.

What we are going to show here is that the generalized minimal Geršgorin set is, in a way, the *best possible localization area* for such matrix pairs. We start by introducing the equimodular set of matrix pairs $\Omega(A, B)$ and the extended equimodular set $\widehat{\Omega}(A, B)$, as treated similarly in [1, Chapter 4].

$$\Omega(A, B) := \left\{(\tilde{A}, \tilde{B}) : |\tilde{a}_{ij}| = |a_{ij}|, |\tilde{b}_{ij}| = |b_{ij}| \text{ and if } b_{ij}\neq 0, \quad \frac{\tilde{a}_{ij}}{\tilde{b}_{ij}} = \frac{a_{ij}}{b_{ij}} \quad i, j\in N\right\} \tag{33}$$

$$\widehat{\Omega}(A, B) := \left\{(\tilde{A}, \tilde{B}) : |\tilde{a}_{ij}| \leqslant |a_{ij}|, |\tilde{b}_{ij}| \leqslant |b_{ij}| \text{ and if } b_{ij}\neq 0, \quad \frac{\tilde{a}_{ij}}{\tilde{b}_{ij}} = \frac{a_{ij}}{b_{ij}} \quad i, j\in N\right\} \tag{34}$$

Now, as is natural, we take the spectrum of these sets to be the union of all the spectra of their elements:

$$\sigma(\Omega(A, B)) := \bigcup_{(\tilde{A}, \tilde{B})\in\Omega(A, B)} \sigma(\tilde{A}, \tilde{B}) \quad \text{and} \quad \sigma(\widehat{\Omega}(A, B)) := \bigcup_{(\tilde{A}, \tilde{B})\in\widehat{\Omega}(A, B)} \sigma(\tilde{A}, \tilde{B}) \tag{35}$$

It is evident from their definitions that

$$\sigma(\Omega(A, B)) \subseteq \sigma(\widehat{\Omega}(A, B)) \subseteq \Gamma^{\mathfrak{R}}(A, B) \tag{36}$$

How tight these inclusions are, is described by the next two theorems.

Theorem 15
For any pair of matrices (A, B) from $\mathbb{C}^{n,n}$ and given an arbitrary $z\in\mathbb{C}$ such that $v(z)$, of (23), satisfies $v(z) = 0$, there exists a matrix pair $(\tilde{A}, \tilde{B})\in\Omega(A, B)$ such that z is a generalized eigenvalue

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of the matrix pair $(\tilde{A}, \tilde{B})$. Thus, the following inclusions hold:

$$\partial\Gamma^{\mathfrak{R}}(A,B)\subseteq\sigma(\Omega(A,B))\subseteq\sigma(\widehat{\Omega}(A,B))\subseteq\Gamma^{\mathfrak{R}}(A,B) \tag{37}$$

Proof

Let $z\in\mathbb{C}$ be such that $v(z)=0$. Then, from the facts leading to (24), we have that there exists a nonzero $\mathbf{y}=[y_1,\ y_2,\ldots,y_n]^{\mathrm{T}}$, with $\mathbf{y}\geqslant\mathbf{0}$, such that $Q_z\mathbf{y}=\mathbf{0}$, or, equivalently,

$$\sum_{j\in N\backslash\{k\}}|b_{kj}z-a_{kj}|y_j=|b_{kk}z-a_{kk}|y_k \quad \text{for all } k\in N$$

which, according to (17) and (18), for every $k\in N$ can be written as

$$\sum_{j\in\beta(k)\backslash\{k\}}\left|z-\frac{a_{kj}}{b_{kj}}\right||b_{kj}|y_j+\sum_{j\in\overline{\beta}(k)\backslash\{k\}}|a_{kj}|y_j=\left|z-\frac{a_{kk}}{b_{kk}}\right||b_{kk}|y_k \quad \text{when } k\in\beta(k) \tag{38}$$

or

$$\sum_{j\in\beta(k)}\left|z-\frac{a_{kj}}{b_{kj}}\right||b_{kj}|y_j+\sum_{j\in\overline{\beta}(k)\backslash\{k\}}|a_{kj}|y_j=|a_{kk}|y_k \quad \text{otherwise} \tag{39}$$

Now, let the real numbers $\{\phi_{kj}\}_{k,j=1}^n$ satisfy

$$\left|z-\frac{a_{kj}}{b_{kj}}\right|=\left(z-\frac{a_{kj}}{b_{kj}}\right)\mathrm{e}^{\mathrm{i}\phi_{kj}} \tag{40}$$

for each $k\in N$ and each $j\in\beta(k)$. Having these numbers, we define the matrices $\tilde{A}=[\tilde{a}_{kj}]$ and $\tilde{B}=[\tilde{b}_{kj}]$, both in $\mathbb{C}^{n,n}$, by means of

$$\tilde{a}_{kj}:=\begin{cases} a_{kj}b_{kj}^{-1}|b_{kj}|\mathrm{e}^{\mathrm{i}\phi_{kj}}, & j\in\beta(k)\\ |a_{kj}|, & j\in\overline{\beta}(k)\end{cases} \tag{41}$$

and

$$\tilde{b}_{kj}:=\begin{cases} |b_{kj}|\mathrm{e}^{\mathrm{i}\phi_{kj}}, & j\in\beta(k)\\ 0, & j\in\overline{\beta}(k)\end{cases} \tag{42}$$

where $j,k\in N$. After a closer look, we can see that $(\tilde{A},\tilde{B})\in\Omega(A,B)$, so that from (38) and (39) respectively, it follows that

$$\sum_{j\in\beta(k)\backslash\{k\}}\left(z-\frac{\tilde{a}_{kj}}{\tilde{b}_{kj}}\right)\tilde{b}_{kj}y_j+\sum_{j\in\overline{\beta}(k)\backslash\{k\}}\tilde{a}_{kj}y_j=\left(z-\frac{\tilde{a}_{kk}}{\tilde{b}_{kk}}\right)\tilde{b}_{kk}y_k \quad \text{when } k\in\beta(k)$$

and

$$\sum_{j\in\beta(k)}\left(z-\frac{\tilde{a}_{kj}}{\tilde{b}_{kj}}\right)\tilde{b}_{kj}y_j+\sum_{j\in\overline{\beta}(k)\backslash\{k\}}\tilde{a}_{kj}y_j=\tilde{a}_{kk}y_k \quad \text{otherwise}$$

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This leads us to the conclusion that $(A-zB)\mathbf{y}=\mathbf{0}$, i.e. $A\mathbf{y}=zB\mathbf{y}$. Thus, z is a (generalized) eigenvalue of a matrix pair $(\tilde{A}, \tilde{B})$, and consequently, from (35), $z\in\sigma(\Omega(A, B))$. □

As the first inequalities in (36) and (37) turn out to be *equalities* for the usual minimal Geršgorin sets (see Theorem 4.5 of [1]), the same is true here.

Theorem 16
For any pair of matrices (A, B) from $\mathbb{C}^{n,n}$,

$$\sigma(\widehat{\Omega}(A, B))=\Gamma^{\mathfrak{R}}(A, B) \tag{43}$$

Proof
Let z be any point of $\Gamma^{\mathfrak{R}}(A, B)$. Then, $v(z)\geqslant 0$, and, from (23), we have that there exists a nonzero vector $\mathbf{y}\in\mathbb{R}^n$, with $\mathbf{y}\geqslant\mathbf{0}$, such that $Q_z\mathbf{y}=v(z)\mathbf{y}$. Writing the last expression by components, we have

$$\sum_{j\in N\setminus\{k\}} |b_{kj}z-a_{kj}|y_j=(|b_{kk}z-a_{kk}|+v(z))y_k \quad \text{for all } k\in N \tag{44}$$

Now, we define real numbers $\{\delta_k\}_{k=1}^n$ as

$$\delta_k:=\begin{cases} \dfrac{\sum_{j\in N\setminus\{k\}}|b_{kj}z-a_{kj}|y_j-v(z)y_k}{\sum_{j\in N\setminus\{k\}}|b_{kj}z-a_{kj}|y_j} & \text{if } \displaystyle\sum_{j\in N\setminus\{k\}}|b_{kj}z-a_{kj}|y_j>0 \\ 1 & \text{if } \displaystyle\sum_{j\in N\setminus\{k\}}|b_{kj}z-a_{kj}|y_j=0 \end{cases} \tag{45}$$

Obviously, from (44), (45) and the fact that $v(z)y_k\geqslant 0$ for each $k\in N$, it follows that $0\leqslant\delta_k\leqslant 1$, and we can construct matrices $\tilde{A}=[\tilde{a}_{jk}]$ and $\tilde{B}=[\tilde{b}_{jk}]$ such that $(\tilde{A}, \tilde{B})\in\widehat{\Omega}(A, B)$, in the following way: for all $k\in N$, $\tilde{a}_{kk}=a_{kk}$ and $\tilde{b}_{kk}=b_{kk}$, while for every for $j\in N\setminus\{k\}$, $\tilde{a}_{kj}=\delta_k a_{kj}$ and $\tilde{b}_{kj}=\delta_k b_{kj}$.

Now, it is readily verified that

$$\begin{aligned} |\tilde{b}_{kk}z-\tilde{a}_{kk}|y_k &= |b_{kk}z-a_{kk}|y_k=\sum_{j\in N\setminus\{k\}}|b_{kj}z-a_{kj}|y_j-v(z)y_k=\delta_k\sum_{j\in N\setminus\{k\}}|b_{kj}z-a_{kj}|y_j \\ &= \sum_{j\in N\setminus\{k\}}|\tilde{b}_{kj}z-\tilde{a}_{kj}|y_j \quad \text{for all } k\in N \end{aligned}$$

which is the same as the starting point of the proof of Theorem 15. As before, we can proceed and can obtain the pair of matrices $(\hat{A}, \hat{B})\in\Omega(\tilde{A}, \tilde{B})$, such that $z\in\sigma(\hat{A}, \hat{B})$. Now, it similarly follows that $z\in\widehat{\Omega}(A, B)$, which completes the proof. □

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REFERENCES

1. Varga RS. *Geršgorin and His Circles*. Springer: Berlin, 2004.
2. Taussky O. Bounds for the characteristic roots of matrices. *Duke Mathematical Journal* 1948; **15**:1043–1044.
3. Taussky O. A recurring theorem on determinants. *American Mathematical Monthly* 1949; **56**:672–676.

4. Cvetković L. H-matrix theory vs. eigenvalue localization. *Numerical Algorithms* 2006; **42**:229–245.
5. Berman A, Plemmons RJ. *Nonnegative Matrices in the Mathematical Sciences*. SIAM: Philadelphia, 1994.
6. Varga RS. Minimal Geršgorin sets. *Pacific Journal of Mathematics* 1965; **15**:719–729.
7. Varga RS, Cvetković L, Kostić V. Approximation of the minimal Geršgorin set of a square complex matrix. *Electronic Transactions on Numerical Analysis* 2008; **30**:398–405.
8. Varga RS. *Matrix Iterative Analysis* (second revised and expanded edition). Springer: Berlin, 2000.

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