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Application of generalized diagonal dominance in wireless sensor network optimization problems

Lj. Cvetković, V. Kostić *

Department of Mathematics and Informatics, Faculty of Science, University of Novi Sad, Serbia

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ABSTRACT

The recent application of the diagonal dominance in the development of the optimization algorithms in the wireless sensor networks design, has been done by Yuan and Yu (2006) [14], extended in Yu et al. (2006) [9], and surveyed in Machado and Tekinay [11]. In this paper, we will use the concept of generalized diagonal dominance, to improve the obtained results regarding the power control game, in three directions. We also discuss the applicability of such improvements.

1. Introduction

A typical wireless sensor network consists of a large number of sensors, which make local measurements of the observed phenomenon, quantify the data, and transfer it back to a central estimation office. Such wireless sensor networks occur in a wide range of applications, where the main challenge is to overcome the overall data distortion that originate from the shared transmission medium, and to use the power resources in the most efficient way. Usually, this is modeled as a network optimization problem. For more details, see [9,11].

In [13], Yuan and Yu decomposed this optimization problem into two disjoint subproblems: a power control subproblem at the physical layer, and a source coding subproblem at the application layer. In their consequent paper [14], they adopted a game theoretic approach to solve each subproblem.

In this paper we will focus on the power allocation optimization problem at the physical layer, and improve one of the main results of [14] in three directions:

- We generalize sufficient conditions under which both games from [14] have a unique and stable Nash equilibrium.
- We introduce more freedom in the management of the power resources, by modifying the original power control game.
- We introduce modified version of the power control algorithm, proposed in [14], which behaves better in certain network settings.

2. Preliminaries

In order to make this paper more easy to read, we will repeat the main setting of the wireless sensor network optimization, given in [14]. In fact, this section should be considered as a slightly modified quotation of Section 1 from [14].

In a wireless sensor network, the design goal is to minimize the total distortion, by jointly optimizing source coding and power allocation, which can be formulated as follows:

* Corresponding author.
E-mail addresses: lila@dmi.uns.ac.rs (Lj. Cvetković), vkostic@dmi.uns.ac.rs, vladimir.slk@gmail.com (V. Kostić).

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$$\begin{aligned} &\text{minimize} \quad \boldsymbol{\alpha}^T d \\ &\text{subject to} \quad s \in \mathcal{R}(d), \quad c \in \mathcal{C}(p), \quad Ac \geqslant s, \end{aligned} \tag{1}$$

where $\boldsymbol{\alpha}$ is a vector, representing the relative emphasis on different elements of the distortion vector d, s is a set of source rates at each node, c is a set of link capacities, and p is the power consumption vector. $\mathcal{R}(d)$ is a fundamental concept in source coding, called the rate-distortion region. The constraint $s \in \mathcal{R}(d)$ models the inter-dependence of the distortion on the source rates. $\mathcal{C}(p)$ is a fundamental concept in channel coding, called the capacity region. The constraint $c \in \mathcal{C}(p)$ models the inter-dependence of the link capacity vector on the power consumption. The last inequality $Ac \geqslant s$ reflects the fact that the source rate at each node must be less than the link capacity support. Here, A is an $m \times n$ node-incident matrix with m nodes and n links, which, using the *multi-commodity flow routing model* [3] can be characterized with:

$$a_{i,j} = \begin{cases} 1, & \text{if } i \text{ is a starting node for the link } j, \\ -1, & \text{if } i \text{ is an end node for the link } j, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Applying the dual decomposition technique [14] the joint optimization problem (1) can be, further, decoupled into two distinct subproblems: a power control subproblem at the physical layer

$$\begin{aligned} &\text{maximize} \quad \boldsymbol{\mu}^T c \\ &\text{subject to} \quad c \in \mathcal{C}(p), \end{aligned} \tag{3}$$

and a source coding subproblem at the application layer

$$\begin{aligned} &\text{minimize} \quad \boldsymbol{\alpha}^T d + \boldsymbol{\lambda}^T s \\ &\text{subject to} \quad s \in \mathcal{R}(d), \end{aligned} \tag{4}$$

where $\boldsymbol{\mu}$ is related to the dual variable $\boldsymbol{\lambda}$ by the *link price consistency equations* $\boldsymbol{\mu}^T = \boldsymbol{\lambda}^T A$. The Lagrange multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ have the interpretation of being the *shadow prices* coordinating the application layer *demand* and the physical layer *supply*.

Now, using the game theory setup, we address the physical layer subproblem that concerns the transmission interference among nearby sensors. Interference management is one of the main challenges in the physical layer design of wireless networks. A key concept at the physical layer is the achievable capacity region, which characterizes a tradeoff between achievable capacities at different links.

We consider a network, where, for every link $i \in N$, $g_{i,i}$, p_i, and ξ_i are the *link gain*, *the power action*, and the *noise of the link*, respectively. By $g_{i,j}$ we denote the *gain of the interference* from link j to link i. The values of the *gain matrix* $G = [g_{i,j}] \in \mathbf{R}^{n,n}$ and the *noise vector* ξ are generally obtained through some estimation techniques, and they characterize the channel[1] statistics. Further, we assume that each node has a certain power budget, such that the power action of the link i is limited by p_i^{max}, i.e., $p \leqslant p^{max} := \left[p_1^{max}, p_2^{max}, \ldots, p_n^{max}\right]^T$. Thus, the power control subproblem (3) with a physical-layer interference model may be formulated as:

$$\begin{aligned} &\text{find} \quad \mathbf{0} \leqslant p \leqslant p^{max} \text{ that maximizes } \sum_{i \in N} \mu_i c_i, \\ &\text{where} \quad c_i := \log(1 + \mathrm{SINR}_i), \\ &\qquad\quad \mathrm{SINR}_i = \frac{g_{i,i} p_i}{\sum_{j \in N \setminus \{i\}} g_{i,j} p_j + \xi_i}, \end{aligned} \tag{5}$$

where c_i is the *capacity* of the link $i \in N$, and SINR_i is its *signal to interference and noise ratio*.

Because of the interference, the power control subproblem (5) is a nonconvex optimization problem that is inherently difficult to solve. The game theory was used to approach this problem iteratively, and solve it. In a power control game that is defined in the sequel, each link is modeled as a player with an aim of maximizing its payoff function. In the conventional game theoretic approaches, each link uses its own achievable rate as the payoff function. Competitive equilibria in such a game may not correspond to desirable operating points, especially when the interference level is high. Thus, the payoff function proposed in [14] is such that each player's (i.e. link's) payoff includes not only its achievable rate, but also the interference effect to other links. Hence, a tax mechanism was introduced into the game, so that the players will have an incentive to intelligently avoid interference, by keeping the signal to interference and noise ratio as high as possible, while, at the same time, tending to minimize the overall power usage.

Mathematically, such a *power control game* consists from the i-th player strategy to maximize its payoff function Q_i, while paying the tax rate t_i, and performing the action p_i. It can be can be expressed as follows:

$$\begin{aligned} &\text{maximize} \quad Q_i := \mu_i \log(1 + \mathrm{SINR}_i) - t_i p_i, \\ &\text{subject to} \quad 0 \leqslant p_i \leqslant p_i^{max}, \end{aligned} \tag{6}$$

where the tax rate for the link $i \in N$, that was proposed in [14], is the rate at which other users' achievable data rates decrease, with an additional amount of power of the link i, i.e.,

[1] For simplicity, we consider the case when each link has one channel. The realistic case can be modeled in the same way.

$$t_i := \left|\frac{\partial \sum_{k\in N\setminus\{i\}} \mu_k c_k}{\partial p_i}\right| = \sum_{k\in N\setminus\{i\}} \frac{\mu_k g_{k,i}}{g_{k,k} p_k \mathrm{SINR}_k^{-1}\left(1+\mathrm{SINR}_k^{-1}\right)}. \tag{7}$$

Here, more power link i uses, more interference it will produce to others, and, therefore, more tax (i.e., $t_i p_i$) it has to pay.

The power vector p that solves the optimization problem (5) is, exactly, the Nash equilibrium of the power control game (6). Since, in general, not every game has a Nash equilibrium, and neither is the equilibrium necessarily stable, at first, the goal is to prove the existence, uniqueness and stability of the Nash equilibrium for the power control game. Then, the aim is to design a distributed iterative algorithm that will converge to that equilibrium.

In [14], a power control game algorithm is proposed. It consists of two phases: the power update, and the tax update. The power update is based upon the fact that, at each step, every player $i \in N$ tries to maximize its own payoff Q_i, while assuming that the power levels of all other players and the taxes are fixed. The expression for such an optimal p_i^* is, then, obtained by setting the derivative Q_i with respect to p_i to zero, i.e., $\frac{\partial Q_i}{\partial p_i} = 0$, and it is called the *best response function* of the player i, denoted by $B_i(p)$. In such a way we have obtained a locally optimal power vector p^*, with the property that for every $i \in N$, p_i^* strikes a balance between maximizing its own rate and minimizing its interference to other links (which is taken into account via t_i). For example, a large value of tax rate t_i indicates that the link i is producing severe interference to other links. This is reflected in the power update, as the larger t_i leads to a lower p_i. Although each player appears to be selfish in maximizing only its own payoff, since the payoff function incorporates social welfare, the Nash equilibrium of this game is, in fact, a *cooperative social optimum*.

Therefore, calculating the locally optimal power vector p^* consists in solving the system of equations $\frac{\partial Q_i}{\partial p_i} = 0$, for $i \in N$, which can be expressed in an equivalent form as $p = B(p) := [B_1(p), B_2(p), \ldots, B_n(p)]^T$. Thus, p^* can be seen as a fixed point of the best response vector function. This approach was used in [14] to obtain the existence, uniqueness and the dynamical stability of the power control game. Here, we will write the system $\frac{\partial Q_i}{\partial p_i} = 0$, for $i \in N$, in a matrix form:

$$Gp = D_G D_t^{-1} \mu - \xi, \tag{8}$$

where $G = [g_{i,j}] \in \mathbf{R}^{n,n}$ is the gain matrix of the links in the wireless network, $D_G := diag(g_{1,1}, g_{2,2}, \ldots, g_{n,n})$ its diagonal part, $D_t := diag(t_1, t_2, \ldots, t_n)$ diagonal matrix of the tax rates, $\xi \in \mathbf{R}^n$ the noise vector, and $\mu \in \mathbf{R}^n$ is the vector of dual variables. This formulation of the problem will alow us to generalize the work of Yuan and Yu in the next section.

Once locally optimal power vector is obtained, the algorithm proceeds with the tax rate update, using the formula (7). As it was given in [14], tax rate can be expressed through the signal to noise ratios in the form that is convenient for the *distributed implementation*. Here distributed implementation signifies that the tax rate update is directly calculated from the information that is received through each individual link. Namely,

$$t_i = \sum_{j\in N\setminus\{i\}} g_{i,j} b_j, \quad (i \in N), \tag{9}$$

where

$$b_i = \mu_i \frac{\mathrm{SINR}_i}{g_{i,i} p_i} \frac{\mathrm{SINR}_i}{1+\mathrm{SINR}_i} \tag{10}$$

is the broadcast message of the link $i \in N$.

Although it is clear that the tax rate vector t is calculated from the actual power vector p, in each power update step this vector is fixed, and therefore the system (8) is the system of linear equations with the system matrix G.

Finally, we give the power control game algorithm from [14] in an equivalent form, more suitable for the discussion given in the last section.

2.1. Power control game algorithm

1. Initialize $p^{(0)}$ and $t^{(0)}$, and set $l = 0$.
2. Iteratively determine p^*, such that

$$Gp = D_G D_{t^{(l)}}^{-1} \mu - \xi,$$

where $D_{t^{(l)}} := diag\left(t_1^{(l)}, t_2^{(l)}, \ldots, t_n^{(l)}\right)$, and set $p^{(l+1)} := p^*$.

3. For each link $i \in N$, calculate signal to noise ratio

$$\mathrm{SINR}_i^{(l+1)} = \frac{g_{i,i} p_i^{(l+1)}}{\sum_{j\in N\setminus\{i\}} g_{i,j} p_j^{(l+1)} + \xi_i},$$

and the broadcast message

$$b_i^{(l+1)} = \mu_i \frac{\mathrm{SINR}_i^{(l+1)}}{g_{i,i} p_i^{(l+1)}} \frac{\mathrm{SINR}_i^{(l+1)}}{1+\mathrm{SINR}_i^{(l+1)}}.$$

4. For each link $i \in N$, update the tax rate

$$t_i^{(l+1)} = \sum_{j \in N \setminus \{i\}} g_{i,j} b_j^{(l+1)}$$

5. Set $l := l + 1$, and return to step 2. until convergence.

3. Generalized diagonal dominance

Strict diagonal dominance is a matrix property that was extensively investigated during the last century. It's main applications were in determining nonsingularity of matrices, in development of various eigenvalue localization techniques, in obtaining convergence results for matrix iterative methods and many others. During the years, many generalizations of strictly diagonally dominant (SDD) matrices were obtained. In this section, we will briefly recall some of them.

Given an arbitrary matrix $A = [a_{i,j}] \in \mathbf{C}^{n,n}$, we denote the set of indices by $N := \{1, 2, \ldots, n\}$, and

$$r_i(A) := \sum_{j \in N \setminus \{i\}} |a_{i,j}| \quad (i \in N) \tag{11}$$

is called the ***i*-th deleted absolute row sum**.

Theorem 3.1. *Let $A = [a_{i,j}] \in \mathbf{C}^{n,n}$ be an arbitrary matrix. If*

$$|a_{i,i}| > r_i(A) \quad \text{for all } i \in N, \tag{12}$$

then A is nonsingular.

Matrices that satisfy the condition (12) are called **strictly diagonally dominant matrices**, or, briefly, **SDD** matrices. Their beauty lies in the fact that, while it takes a lot of computation to determine whether $det(A)$ is equal to zero or not, the condition (12) is rather easy to verify. Nonsingularity of SDD matrices is, in fact, a starting point for many other interesting results. Definitely, one of the most famous is *Geršgorin theorem* about the region of complex plane that contains eigenvalues of a given matrix.

Given an arbitrary square matrix $A = [a_{i,j}] \in \mathbf{C}^{n,n}$, the set of its eigenvalues is called the **spectrum**, and is denoted by $\sigma(A)$, i.e.,

$$\sigma(A) := \{\lambda \in \mathbf{C} : det(\lambda I_n - A) = 0\}. \tag{13}$$

Additionally, we define

$$\begin{cases} \Gamma_i(A) := \{z \in \mathbf{C} : |z - a_{i,i}| \leqslant r_i(A)\}, \quad (i \in N), \\ \Gamma(A) := \bigcup_{i \in N} \Gamma_i(A). \end{cases} \tag{14}$$

Theorem 3.2. (Geršgorin theorem) *Given an arbitrary matrix $A = [a_{i,j}] \in \mathbf{C}^{n,n}$, let λ be its eigenvalue. Then, there exists an index $k \in N$, such that*

$$|\lambda - a_{k,k}| \leqslant r_k(A), \tag{15}$$

implying that $\lambda \in \Gamma_k(A)$, and, therefore $\lambda \in \Gamma(A)$. Since $\lambda \in \sigma(A)$ is arbitrary, consequently, it follows that

$$\sigma(A) \subseteq \Gamma(A). \tag{16}$$

Using these two very well known results, Yuan and Yu obtained the main result of [14]. But, in the theory of matrices, one can find another, also very well known, class of nonsingular matrices that can be used for the same purpose.

Definition 3.3. Given an arbitrary matrix $A = [a_{i,j}] \in \mathbf{C}^{n,n}$, if there exists an (entrywise) positive vector $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T \in \mathbf{R}^n$, such that AX is strictly diagonally dominant matrix, where $X := diag(x_1, x_2, \ldots, x_n)$, then, the matrix A is **generalized diagonally dominant**, or, briefly, a **GDD matrix**. Matrix X used above, is called a **scaling matrix**.

Matrices of the previous definition are also known in the literature as nonsingular H-matrices, for more details see [2,5,6].

4. Improvement of the stability criteria for power control game

In [14], one of the main results is that, under the condition that the gain matrix G is an SDD matrix, the power control game is asymptotically stable, and it (more precisely, its power control game algorithm) always converges to the unique Nash equilibrium. But, through the simulations, the authors have noticed that, even if this condition is *not* satisfied, the power control game could converge nicely. Namely, although it may seam as a natural condition, SDD property of the overall gain matrix G could be ruined, due to the stronger interferences inherent to the link topology, or to the state of the medium, through which the carrier wave is propagated. But, sometimes, while the augmented interference rates, that come from the specific links, are such that, for some link $i \in N$, the link gain $g_{i,i}$ is dominated by the interferences from other links $\sum_{j \in N \setminus \{i\}} g_{i,j}$, these interferences will not cross the point at which the power control algorithm fails. Therefore, a natural question is whether we can improve known theoretical results, in order to guarantee the convergence and the stability of the power control game, and its iterative procedure, in wider range of real situations. So, here we formulate the generalization of the Yuan-You's theorem on the stability and the convergence of the power control game.

Theorem 4.1. *Given a wireless sensor network with certain link topology, let $G = [g_{i,j}] \in \mathbf{R}^{n,n}$, $G \geqslant O$, be the overall gain matrix, and $\xi \in \mathbf{R}^n$, $\xi \geqslant \mathbf{0}$, the overall link noise vector. If G is a generalized diagonally dominant matrix, then the power control game, given by (6), where the tax rate vector t is defined by (7), has a unique stable Nash equilibrium p^*. Moreover, the game is asymptotically stable, and the power control game algorithm converges to p^*, for each starting nonnegative vectors $p^{(0)}, t^{(0)} \in \mathbf{R}^n$.*

Proof. First, for every $i \in N$, the payoff function Q_i, given in (6), of the link (player) i is continuous in p, and strictly concave in p_i (which can be verified by computing its Hessian). Hence, since the i-th link action profile $[0, p_i^{max}]$ is a compact convex set, by Theorem 4.3 in [1], it follows that the power control game has at least one pure Nash equilibrium, which can be found as an intersection point of the reaction curves of all the players. Namely, if by p^* we denote Nash equilibrium of the game (6), p^* satisfies the system of linear Eq. (8), i.e., $Gp^* = D_G D_t^{-1} \mu - \xi$. But, since G is a GDD matrix, it is nonsingular, and, thus, $p^* = G^{-1}(D_G D_t^{-1} \mu - \xi)$ is the unique Nash equilibrium of the power control game (6).

In order to prove that this (local) Nash equilibrium is stable, as in [14], we prove the asymptotic stability of the game (6). We will use the concept of the best response function $B(p)$, and the *dynamic stability matrix* $\Delta := [\Delta_{i,j}]$, where $\Delta_{i,j} := \frac{\partial B_i(p)}{\partial p_j}$, for $i,j \in N$. According to [8], the game is asymptotically stable if all the eigenvalues of the dynamic stability matrix lie in the unit circle, i.e., if $\rho(\Delta) < 1$.

In our case, for $i \in N$, the best response of the link i is:

$$B_i(p) = \frac{\mu_i}{t_i} - \frac{1}{g_{i,i}} \left(\sum_{j \in N \setminus \{i\}} g_{i,j} p_j + \xi_i \right),$$

and, thus,

$$\Delta i,j = \frac{\partial B_i(p)}{\partial p_j} = -\frac{g_{i,j}}{g_{i,i}},$$

for $j \in N \setminus \{i\}$, while $\Delta_{i,i} = 0$.

Now, since $G \geqslant O$ is a GDD matrix, there exists a positive diagonal matrix $X = diag(x_1, x_2, \ldots, x_n)$, such that GX is an SDD matrix, i.e.,

$$g_{i,i} x_i > \sum_{j \in N \setminus \{i\}} g_{i,j} x_j, \quad (i \in N),$$

or, equivalently,

$$r_i(X^{-1} \Delta X) = \sum_{j \in N \setminus \{i\}} \frac{g_{i,j} x_j}{g_{i,i} x_i} < 1, \quad (i \in N).$$

On the other hand,

$$\sigma(\Delta) \subseteq \Gamma(X^{-1} \Delta X) = \bigcup_{i \in N} \Gamma_i(X^{-1} \Delta X).$$

So, for every eigenvalue $\lambda \in \sigma(\Delta) = \sigma(X^{-1} \Delta X)$, there exists $i \in N$, such that

$$|\lambda - \Delta_{i,i}| \leqslant r_i(\Delta),$$

and consequently $|\lambda| < 1$. To complete the proof, we observe that the sequence of tax rates is convergent, and therefore, the power control algorithm converges for every starting vectors $p^{(0)}, t^{(0)} \in \mathbf{R}^n$. □

Before we continue, it is interesting to note that the topology of the wireless network can lead to specific structure of the matrix G. Namely, knowing that the interferences between the links occur if the links are "close" to each other, for certain network topologies we can have specific patterns of matrix entries. Therefore, matrix properties, like block forms and reducibility, could be used, in order to obtain different improvements in modeling wireless sensor networks.

Although it seams rather difficult to check if a given matrix is GDD, comparing to the property SDD, one can find many generalizations of SDD matrices in the form of subclasses of GDD matrices which are not so difficult to check. Many of these classes of matrices can be found in [5,6,10,12]. Here we will emphasize one of them, S-SDD matrices, since it can be used to improve the optimization result in the specific network settings.

Theorem 4.2 [10]. *Let $A = [a_{i,j}] \in \mathbf{C}^{n,n}$, with $n \geqslant 2$, be an arbitrary matrix, and let $S \subseteq N$ be a nonempty subset of indices. If, for every two indices $i \in S$, and $j \in \overline{S} := N \setminus S$,*

$$|a_{i,i}| > r_i^S(A), \quad \text{and} \tag{17}$$

$$\left(|a_{i,i}| - r_i^S(A)\right) \cdot \left(|a_{j,j}| - r_j^{\overline{S}}(A)\right) > r_i^{\overline{S}}(A) r_j^S(A), \tag{18}$$

holds, where $r_i^S(A) := \sum_{j \in S \setminus \{i\}} |a_{i,j}|$, then A is nonsingular.

In the literature, matrices defined in the previous theorem are called **S-strictly diagonally dominant** matrices, or just **S-SDD** matrices, where S is the fixed set of indices for which the conditions (17) and (18) hold.

Actually, for this class of matrices, form of the scaling matrix is explicitly known, for more details see [7], and this will be the main point of our next improvement.

5. Optimizing the power deficient network nodes

Another interesting application of the generalized diagonal dominance lies in the usage of S-SDD matrices, of Theorem 4.2, and the underlying scaling technique. Namely, if the overall gain matrix G is an SDD matrix, it is an S-SDD matrix, too, for an arbitrary set of links $S \subseteq N$. Therefore, we can use the information contained in the adequate scaling matrix X, in order to introduce more freedom in the management of the power resources. Namely, given the multi-hop wireless sensor network with the SDD overall gain matrix G, assume that several nodes work with the severe power constraints, but due to the their location in the network topology, they have to be deployed for measuring and/or transmitting. In this case, we would like to prolong the life time of such relays, and the overall optimization of such network should *additionally* minimize the power action of such links, while achieving the Nash equilibrium of the power control game, which maximizes the overall network capacity. To address this issue, we will use the scaling technique developed in [7].

First, let M_0 be the set of nodes that are having restrictive power consumption. Having the node-link incidence matrix $A = [a_{i,j}]$, given by (2), we define the set of power restricted links $L := \{j \in N : a_{i,j} = 1,\ i \in M_0\}$. Since the gain matrix G is an SDD matrix, then it is also an S-SDD, where $S = L$, i.e., meaning that for each $i \in L$, and every $j \in \overline{L} := N \setminus L$,

$$(g_{i,i} - r_i^L(G))(g_{j,j} - r_j^{\overline{L}}(G)) > r_i^{\overline{L}}(G) r_j^L(G), \quad \text{and}$$
$$g_{i,i} > r_i^L(G),$$

where $r_i^L(A) := \sum_{j \in L \setminus \{i\}} g_{i,j}$. But, defining the quantities $\alpha_L(G)$ and $\beta_L(G)$, as follows

$$\alpha_L(G) := \min_{i \in L} \frac{r_i^{\overline{L}}(G)}{(g_{i,i} - r_i^L(G))}$$

and

$$\beta_L(G) := \max_{j \in \overline{L}, r_j^L(G) \neq 0} \frac{g_{j,j} - r_j^{\overline{L}}(G)}{r_j^L(G)},$$

as suggested in [7], we can see that

$$0 \leqslant \alpha_L(G) < 1 < \beta_L(G),$$

and that for each $\gamma \in (\alpha_L(G), \beta_L(G))$, the matrix $\widetilde{G} := GX$ is SDD, where $X = diag(x_1, x_2, \ldots, x_n)$, with

$$x_j = \begin{cases} \gamma, & \text{if } j \in L, \\ 1, & \text{otherwise}. \end{cases}$$

Thus, by setting $\widetilde{p}_i := \frac{p_i}{x_i}$, for $i \in N$, the power control game (6) becomes

$$\begin{aligned} &\text{maximize} && Q_i := \mu_i \log(1 + \widetilde{\text{SINR}}_i) - \widetilde{t}_i \widetilde{p}_i, \\ &\text{by changing} && 0 \leqslant \widetilde{p}_i \leqslant p_i^{max}, \end{aligned} \tag{19}$$

where the tax rate $\widetilde{t}_i := t_i x_i$, for each link $i \in N$, satisfies

$$\widetilde{t}_i := \sum_{k \in N \setminus \{i\}} \frac{\mu_k \widetilde{g}_{k,i}}{\widetilde{g}_{k,k} \widetilde{p}_k \widetilde{\text{SINR}}_k^{-1} \left(1 + \widetilde{\text{SINR}}_k^{-1}\right)}, \tag{20}$$

and $\widetilde{\text{SINR}}_i := \frac{\widetilde{g}_{i,i} \widetilde{p}_i}{\sum_{j \in N \setminus \{i\}} \widetilde{g}_{i,j} \widetilde{p}_j + \xi_i}$, for $i \in N$.

Now, having that the matrix $\widetilde{G}$ is SDD, using Theorem 4.1, we have that the game (19) is asymptotically stable, and we obtain the unique Nash equilibrium $\widetilde{p}^*$ that satisfies the equality $\widetilde{p}^* = X^{-1}p^*$. Thus,

$$\widetilde{p}_i^* = \begin{cases} \gamma^{-1}p_i^*, & \text{if } i \in L, \\ p_i^*, & \text{otherwise}. \end{cases}$$

The obtained relation shows the connection between the originally obtained vector of the link power action that is Nash equilibrium of the power control game, and the new one. Since the links $i \in L$ are such that it is desirable to have as small as possible power action, we wish to adjust the parameter γ to be bigger than 1, and as big as possible. But, since we have that $1 < \beta_L(G)$, by choosing $1 < \gamma < \beta_L(G)$ to be sufficiently close to the value of $\beta_L(G)$, for each $i \in L$, we have that $\widetilde{p}_i^* < p_i^*$, while, for $i \in \overline{L}$, $\widetilde{p}_i^* = p_i^*$. Therefore, we have obtained the unique Nash equilibrium that better suites the power constraints of the given wireless sensor network.

6. Various versions of the power control algorithm and their convergence

In the sequel, we focus on the power control algorithm (briefly **PCA**), and we are interested to improve its convergence speed. Since one part of the overall energy consumption in wireless sensor network is spent in calculation, used to implement the PCA, the complexity of calculation, and the speed of convergence are issues that should be treated. We start with the observation that PCA consists of the inner iteration and outer iteration. The inner iteration is the power allocation vector update (step 2), performed at each step l. The outer iteration consists of the tax rate update through broadcast message vector. The original algorithm that was given in [14], in step 2 of PCA performed, at each link, a fixed point iteration, using the best response function of the concerned link. Namely, the inner iterative procedure is given by $p^{(k+1)} := B(p^{(k)}) = [B_1(p^{(k)}), B_2(p^{(k)}), \ldots, B_n(p^{(k)})]^T$, for any $p^{(0)}$, and all $l \in \mathbf{N}$. Equivalently, this can be written as:

$$p_i^{(k+1)} := \frac{\mu_i}{t_i^{(l)}} - \frac{1}{g_{i,i}}\left(\sum_{j\in N\setminus\{i\}} g_{i,j}p_j^{(k)} + \xi_i\right), \tag{21}$$

for $i \in N$ and $k \in \mathbf{N}$, where $p^{(0)}$ is arbitrary. When, at some step k_0, the iterative approximation is satisfactory, the link power allocation at that step is, then, forwarded to the outer iteration, i.e., to the tax rate update. The convergence of this procedure was obtained through the argument that the best response function of the link i is a contraction. The described procedure can be seen as distributed one, meaning that the power update for link i is obtained by the calculation that can be implemented using exclusively the information that link i is capable to measure. Therefore, each link is capable to make its own power update, using the actual power consumption vector of the overall network. The similar argument stands for the tax rate update, too. Finally, under the assumption that the gain matrix is SDD, the authors proved the asymptotic convergence of the game, and, thus, the convergence of PCA.

Here we will address only the inner iteration. We propose new iterative procedures, discuss their implementation and convergence. The main idea is based upon the fact that the locally optimal equilibrium p^* can be obtained as the solution of the linear system (8). Under the assumption that the overall gain matrix is an H-matrix, in the previous considerations, we have proven the asymptotic stability of the game (6), and, hence, obtained the convergence of the PCA. Therefore, if we obtain, under the same condition, the convergence of the new procedures for the inner iteration, the modified PCA will also converge to the Nash equilibrium of the power control game (6).

Given wireless network with the certain topology of n links and the overall gain matrix $G = [g_{i,j}] \in \mathbf{R}^{n,n}$, with $D_G = diag(g_{1,1}, g_{2,2}, \ldots, g_{n,n})$, we denote $B_G := G - D_G$. For the fixed tax rates $t = [t_1, t_2, \ldots, t_n]^T$ of the power control game (6), we have, as before, $D_t = diag(t_1, t_2, \ldots, t_n)$. Let, again, $\xi \in \mathbf{R}^n$ be the noise vector, and $\mu \in \mathbf{R}^n$ the vector of dual variables in the power control subproblem (3) of the cross-layer optimization problem (1). If G is an H-matrix, then, the locally optimal vector of the link power allocation p^* is the unique solution of the system of linear equations

$$Gp = D_G D_t^{-1}\mu - \xi. \tag{22}$$

If we use the splitting of the matrix $G = D_G - B_G$, then we can write (22) in the fixed point form $p = D_G^{-1}(B_G p - \xi) + D_t^{-1}\mu$, and define the iteration procedure $p^{(k+1)} = D_G^{-1}(B_G p^{(k)} - \xi) + D_t^{-1}\mu$. Since we took the Jacobi splitting of the system matrix G, the iterative method is the famous **Jacobi iteration**. On the other hand, it is easy to see that this procedure is exactly (21), the one proposed by Yuan and Yu.

The other fundamental procedure in the theory of iterative methods is, off course, Gauss–Seidel iteration scheme. Given a matrix $G = [g_{i,j}]$, consider the standard splitting $G = D_G - L_G - U_G$, where D_G is a diagonal matrix, while L_G and U_G are, respectively, strictly lower and strictly upper triangular matrices. More precisely, let $D = diag(g_{1,1}, g_{2,2}, \ldots, g_{n,n})$, $L = [l_{i,j}]$, where

$$l_{i,j} = \begin{cases} -g_{i,j}, & j < i, \\ 0, & \text{otherwise}, \end{cases}$$

and $U = [u_{i,j}]$, where

$$u_{i,j} = \begin{cases} -g_{i,j}, & j > i, \\ 0, & \text{otherwise}. \end{cases}$$

Then, **Gauss–Seidel** itarative method for the system (22) can be written as:

$$D_G p^{(k+1)} = L_G p^{(k+1)} + U_G p^{(k)} - \xi + D_G D_t^{-1} \mu,$$

for $k \in \mathbf{N}$.

In the form of the power update procedure for each link $i \in N$, we obtain

$$p_i^{(k+1)} := \frac{\mu_i}{t_i^{(l)}} - \frac{1}{g_{i,i}} \left(\sum_{j=1}^{i-1} g_{i,j} p_j^{(k+1)} + \sum_{j=i+1}^{n} g_{i,j} p_j^{(k)} + \xi_i \right), \tag{23}$$

for $i \in N$ and $k \in \mathbf{N}$.

Here, we assume that the link 1 is first to update its power, then link 2, link 3, and, finally, the link n. Since the system matrix is an H-matrix, it is well known that Gauss–Seidel iterative method is globally convergent. But, although there are many examples where the Gauss–Seidel iteration is preferable than the Jacobi iteration, we cannot state that, in general, iterative procedure (23) works faster than (21).

While often performing faster then Jacobi iteration, Gauss–Seidel iteration has, in this case, a significant drawback. Namely, due to the sequentiality, the link that has to update its power has often to wait its turn. But this is not necessary, since each link is updating its power with the data it has already collected. Therefore, the Gauss–Seidel procedure, while in theory good, behaves rather poorly in the wireless sensor networks due to the link's computational standby time in the iterative procedure in the step 2. of PCA. The answer to this drawback of Gauss–Seidel is the **chaotic asynchronous relaxation**, developed by Chazan and Miranker in [4]. Without going into detailed notation, we remark that this algorithm uses the same rule as (21), while the power levels on the right hand side are not necessarily from the same iteration step. Namely, each link uses the most recent powers of other links to update its own. The main value of this algorithm is that, in a wireless sensor network, it behaves very good, in a way that it avoids the link standby time due to asynchronous computations in power update iterations, while it allows the distributed implementation. The only issue that needs to be addressed is the convergence. But, the fundamental theorem on the chaotic asynchronous relaxation states that this iterative method converges if all the eigenvalues of the modulus of the Jacobi iteration matrix lie inside the unit disk, in our case, if $\rho(|D_G^{-1} B_G|) < 1$. But, if the overall gain matrix G is an H-matrix, this is true. To prove it, assume that $\lambda \in \sigma\left(\left|D_G^{-1} B_G\right|\right)$. Since G is an H-matrix, then there exists a positive diagonal matrix $X = diag(x_1, x_2, \ldots, x_n)$, such that GX is an SDD matrix. But, from (16), there exists $i \in N$, such that $\lambda \in \Gamma_i\left(X^{-1}\left|D_G^{-1} B_G\right| X\right)$, and, hence, $|\lambda| \leqslant \sum_{j \in N \setminus \{i\}} \frac{g_{i,j} x_j}{g_{i,i} x_i} < 1$. So, implementing the chaotic asynchronous relaxation procedure in the step 2. of PCA, the algorithm will converge to the unique and stable Nash equilibrium of the power control game.

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